Maximum Likelihood Ratio distribution of a modulated exponential decay *

N. Winckler[1] *and the Two-Body Weak Decay collaboration*[1]
[1]GSI, Darmstadt

Introduction

In this contribution we discuss and determine the distribution of the unbinned maximum likelihood ratio test statistic (LRT) with the hypotheses made in Ref. [1]. The study of this distribution is important to correctly assess the statistical significance or to determine the Type I error rate of Akaike or Bayesian information criteria.

Problems

Given the $i.i.d.$ random variable $t \in I$, and the parameter space $\theta = (\lambda, a, \omega, \phi)$ we define the probability density function :

$$f(t, \theta) = N_{\theta,I}(1 + a\cos(\omega t + \phi))\exp(-\lambda t), \quad (1)$$

where $N_{\theta,I}$ is a normalisation factor depending on θ and the measuring time interval I. We propose to evaluate the LRT distribution, $-2\log(\hat{L_0}/\hat{L_1})$, where $\hat{L_0}$ and $\hat{L_1}$ denote the maximum likelihood of the model $f(t, \theta)$ with $a = 0$ and $a \neq 0$, respectively. There are two connected problems with this hypothesis testing. The first problem is that the hypothesis is a point null hypothesis, which often produce significant results in favour of the alternative hypothesis [2]. The second problem is that, under the null hypothesis, the modulation parameters are *non identifiable parameters*, and in such a case the Wilks's theorem, which says that the asymptotic distribution of the LRT statistic is distributed as χ^2_d, is generally wrong, and the correct asymptotic limit depends very much on the precise problem being investigated [3, 4]. Given the dimension of the identifiable parameter(s) p and the dimension of the non identifiable parameter(s) q, some approaches have shown that an approximation of the asymptotic distribution could be obtain when $p \geq 1$ and $q = 1$ or when $p = 1$ and $q > 1$ [4, 5]. Since in our case we have $p = 2$ and $q = 2$, these approaches cannot be applied. Therefore a Monte Carlo approach has been performed to evaluate this distribution.

Method and Results

In addition to the two problems mentioned above, a third problem arises due to the statistical non-consistency of the maximum likelihood estimate (MLE) of the parameters ω (the first derivative of the likelihood w.r.t. ω is not monotone, and, accordingly, several roots can be found, at least for a finite sample size $N \approx 4000$). Therefore a simple fitting procedure is not reliable and will strongly depends on the initialization of the parameters, the minimization procedure being traped by local maxima.

To evaluate the distribution we first use Monte Carlo toys to simulate the null hypothesis, i.e. a pure exponential decay with the same sample size N as the one in ref. [1]. The maximum likelihood of the two models as well as the corresponding MLE are found by the Metropolis algorithm with an adaptative Breit-Wigner proposal function. This has been achieved using the MCMC engine of the Bayesian Analysis Toolkit [6]. One Markov chain, with a maximum of 10000 iterations, turned out to be more reliable and faster than a maximum likelihood ratio profile. In order to study in addition the pull distributions and bias of each parameters, the (observed) covariance matrix is required. Therefore, the MLEs found by the metropolis algorithm are used to initialize an unbinned maximum likelihood procedure using the roofit package[7].

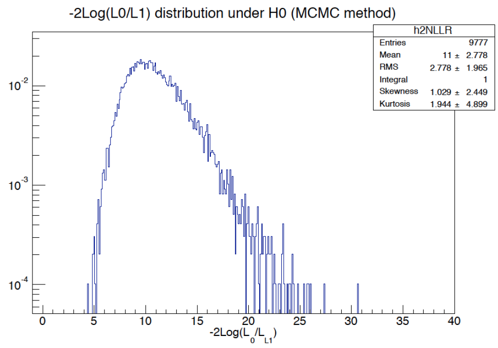

Figure 1: Unbinned maximum likelihood ratio distribution.

Over 10000 iterations of the above procedure about 2% of the MLE were lying on the boundaries. Therefore these MLE have been excluded from the analysis, resulting to a LRT distribution with a sample size of 9777. The resulting distribution is shown in Figure 1 and can be used to test real data with a lower bound $p \geq 1/9777$.

References

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